Math Circles - Pigeonhole Principle - Fall 2022

Exercises

1. For Halloween this year, you decide to adopt a new strategy to give out candy to the trick-or-treaters. The night before Halloween, you put together packages of candy; one package for each of the 42 trick-or-treaters that you expect, plus one for yourself (because of course you want candy too). In order to be nice, you make sure that every package has at least one candy in it, but after that, you randomly distribute the candy amongst the packages. Once the candy has all been distributed, you choose the biggest package and keep it for yourself, and leave the other packages to the trick-or-treaters. If candy comes in boxes of 45 pieces, how many boxes of candy do you need to buy to ensure you end up with at least 7 pieces of candy in your package?

2. Prove that, on an 8×8 chessboard, it is impossible to place nine rooks[1] so that no two rooks threaten each other.

3. Imagine that you're trying to cover an 8×8 chessboard with dominoes, where each domino covers two adjacent squares. It is easy to see that this is possible on a normal chessboard, however is it still possible if you remove two diagonally-opposite corners?

4. Suppose we have five distinct lattice points[2] on the xy-plane. If each pair of points is connected by a line, show that at least one of these lines has a lattice point in its interior.[3]

5. Given nine points inside a unit square,[4] show that three of these points must form a triangle whose area is less than or equal to $\frac{1}{8}$.

6. We are given a square that has 9 lines drawn on it. Each line divides the square into two quadrilaterals such that one quadrilateral contains $\frac{1}{3}$ of the area of the square and the other quadrilateral contains $\frac{2}{3}$ of the area of the square. Prove that at least 3 of these 9 lines pass through the same point.

[1] Recall that rooks can move as far as they want along the column or row that they are in.
[2] A *lattice point* is a point whose coordinates are integers.
[3] The *interior* of a line segment means all points of the line, except for its endpoints.
[4] A *unit square* is a square with side length 1.